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GENERATION OF CRACKS IN A PERFORATED REINFORCED PLATE

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A problem of fracture mechanics on crack nucleation in a reinforced plate attenuated by a periodic system of circular holes is considered. Crack nucleation is modeled by a pre-fracture band in the plastic flow state with a constant stress, which is considered as a region of attenuated bonds between material particles. Determining unknown parameters characterizing the emerging crack reduces to solving a singular integral equation. The condition of crack emergence is formulated with allowance for the criterion of the limiting opening of the faces of the material pre-fracture band.

Key words: *pre-fracture band, crack emergence, perforated reinforced plate, stringers.*

To ensure sufficient strength, sheet structures are usually made of thin planes reinforced by riveted stiffness ribs. Plates used in engineering structures have technological holes, which are stress concentrators and facilitate crack nucleation. To further increase the stiffness, plates are reinforced by stringers. Deformation and fracture of a reinforced plate have been studied in many papers (see, e.g., [1]). At the same time, the study of crack nucleation is also important. Various mechanisms of crack formation are currently known [2–4]. Motion of dislocations can lead to divergence of atomic planes; as a result, force interaction between these planes ceases to exist. Slot-like voids formed thereby can be treated as microcracks with sizes of the order of the characteristic lattice size. Modeling the processes of crack emergence and development is a complicated problem, first of all, from the physical and technical points of view.

Formulation of the Problem. An elastic isotropic thin plate attenuated by a periodic system of circular holes of radius λ is considered. Transverse stiffness ribs made of another elastic material with a cross-sectional area A_s are riveted to the plate at points $z = \pm(2m+1)L \pm iny_0$, where $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, (see Fig. 1). At infinity, the plate is subjected to uniform tension along the stringer under a stress $\sigma_y^\infty = \sigma_0$. A hypothesis about a one-dimensional continuum is accepted for the stringer. The stringers do not exert any resistance to bending and works in tension only. The action of the stringers is modeled in the calculation scheme by unknown equivalent point forces applied at the points where the ribs are connected to the plate. As the intensity of external loading is increased, zones with elevated stresses are formed around the holes in the reinforced plate; these zones are located periodically. The presence of elevated stress zones favors the emergence of edge cracks.

When a riveted plate is loaded by certain forcing, there arise pre-fracture zones, which are modeled as regions with attenuated bonds of material particles. The pre-fracture band is assumed to be aligned in the direction perpendicular to the action of the maximum tensile stresses emerging in the reinforced plate. The problem of the stress-strain state of a deformable solid with interlayers of a "superstressed" material inside the body can be reduced to the problem of the stress-strain state in an elastic body attenuated by cuts whose surfaces interact in accordance with a certain law. In such an approach to solving the problem of nucleation of a crack-type defect, the primary necessity is to find the dependence between the forces and displacements in the region of the deformable material where interparticle interaction forces act (attraction of the faces). Taking these effects into account in problems of fracture mechanics of machines and structures is an important but simultaneously a difficult problem.

The reinforced plate is modeled by an elastic (brittle) solid. In the course of deformation, zones where Hooke's law is not satisfied can appear at certain points of the plate, i.e., the stresses in these zones are higher than

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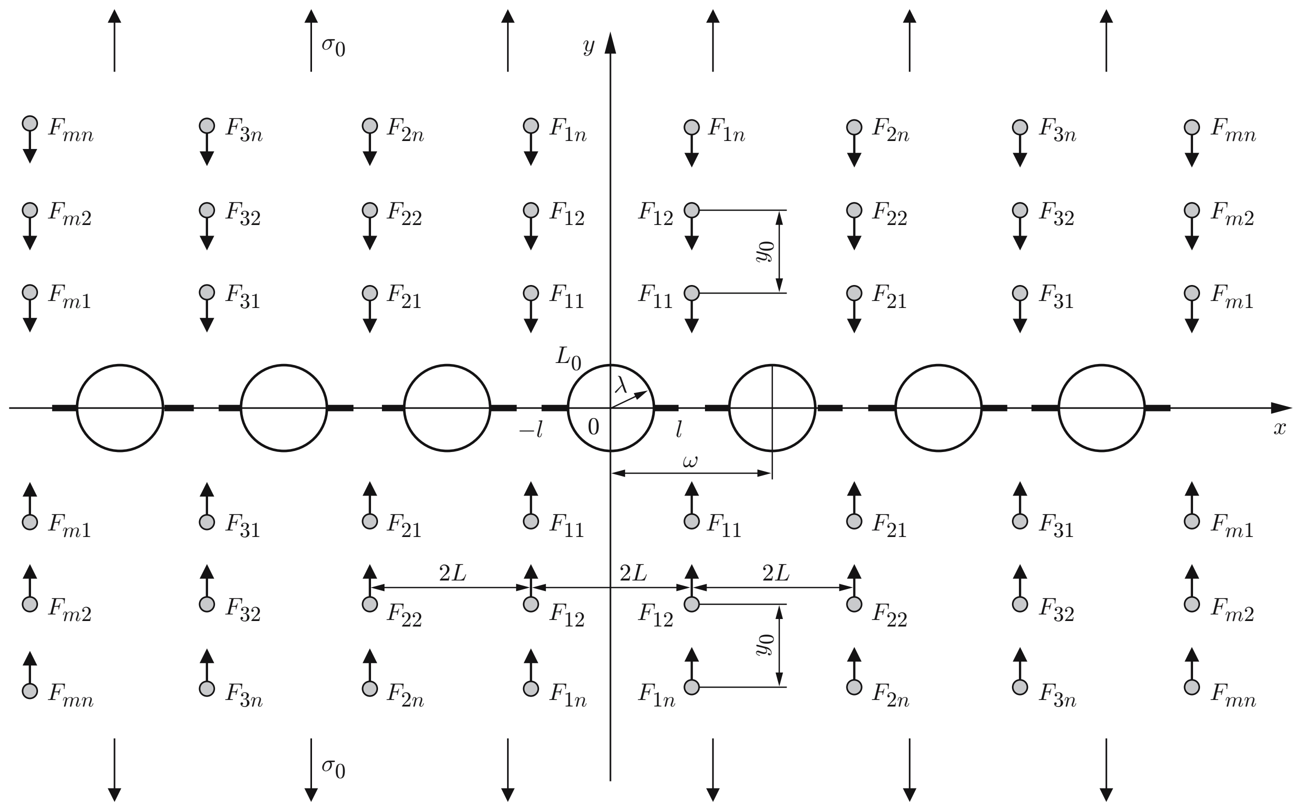

Fig. 1. Calculation scheme of the problem of crack nucleation in a perforated reinforced plate.

the yield point. As these zones (interlayers) are small, as compared with the remaining elastic part of the plate, they can be replaced by cuts whose surfaces interact in accordance with a certain law corresponding to the action of the removed material. The size of the pre-fracture band is not known in advance and has to be determined in the course of solving the problem.

Experimental studies of the emergence of zones with the material being deformed beyond the elasticity limit show that the pre-fracture zones at the initial stage form a narrow extended layer; as the load is increased, a secondary system of plastic strain zones appears suddenly [2, 5–7].

Let us consider a reinforced plate with a periodic system of circular holes of radius λ ($\lambda < 1$) with the hole centers being located at the points

$$P_m = m\omega, \quad m = \pm 1, \pm 2, \dots, \qquad \omega = 2.$$

Symmetric straight-line pre-fracture bands emanate from the hole contours (see Fig. 1). We assume that there is a plastic flow at a constant stress in the pre-fracture band.

The boundary conditions in the pre-fracture band have the form

$$\sigma_y = \sigma_{\text{yield}}, \qquad \tau_{xy} = 0, \tag{1}$$

where σ_{yield} is the yield stress of the plate material in tension.

The contours of the circular holes are free from external forces. To determine the applied load responsible for crack emergence, the formulation of the problem should be supplemented by the condition (criterion) of crack emergence (breakdown of interparticle bonds of the material). As such a condition, we use the criterion of the critical opening of the pre-fracture band faces

$$v^+(x,0) - v^-(x,0) = \delta_c, \tag{2}$$

where δ_c is the characteristic of material resistance to cracking, which is determined in experiments; the superscripts plus and minus refer to the boundary values on the upper and lower faces of the pre-fracture band.

The additional condition (2) allows us to determine the parameters of the reinforced plate at which the crack emerges.

On the basis of the Kolosov–Muskhelishvili formulas [8] and the boundary conditions on the contours of the circular holes L_m ($m = 0, \pm 1, \pm 2, \ldots$) and pre-fracture band faces [see Eq.(1)], the problem is reduced to determining two analytical functions $\Phi(z)$ and $\Psi(z)$ from the boundary conditions

$$\Phi(\tau) + \overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\, e^{2i\theta} = 0; \tag{3}$$

$$\Phi(x) + \overline{\Phi(x)} + x\overline{\Phi'(x)} + \overline{\Psi(x)} = \sigma_{\text{yield}}, \tag{4}$$

where $\tau = \lambda\, e^{i\theta} + m\omega$ ($m = 0, \pm 1, \ldots$) and x is the affix of the points of the pre-fracture band faces.

Solution of the Boundary-Value Problem. We seek for the solution of the boundary-value problem (3), (4) in the form

$$\Phi(z) = \Phi_0(z) + \Phi_1(z) + \Phi_2(z), \qquad \Psi(z) = \Psi_0(z) + \Psi_1(z) + \Psi_2(z), \tag{5}$$

where the potentials $\Phi_0(z)$ and $\Psi_0(z)$ determine the stress and strain fields, respectively, in the reinforced plate without pre-fracture bands under the action of the point forces F_{mn} and σ_0

$$\Phi_0(z) = \frac{1}{4}\,\sigma_0 - \frac{i}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{1}{z - mL + iny_0} - \frac{1}{z - mL - iny_0}\Big),$$

$$\Psi_0(z) = \frac{1}{2}\,\sigma_0 - \frac{i\varkappa_0}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{1}{z - mL + iny_0} - \frac{1}{z - mL - iny_0}\Big) \tag{6}$$

$$+ \frac{i}{2\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{Lm - iny_0}{(z - mL - iny_0)^2} - \frac{mL + iny_0}{(z - mL + iny_0)^2}\Big),$$

where h is the plate thickness, $\varkappa_0 = (3-\nu)/(1+\nu)$, and ν is Poisson's ratio of the plate material; the prime at the summation sign indicates that the subscript $m = n = 0$ is eliminated in summation.

The functions $\Phi_1(z)$ and $\Psi_1(z)$ corresponding to unknown normal displacements along the pre-fracture bands are sought in an explicit form as

$$\Phi_1(z) = \frac{1}{2\omega}\int_{L_1^*} g(t)\cot\frac{\pi}{\omega}(t-z)\,dt, \qquad \Psi_1(z) = -\frac{\pi z}{2\omega^2}\int_{L_1^*} g(t)\sin^{-2}\frac{\pi}{\omega}(t-z)\,dt,$$

$$L_1^* = [-l, -\lambda] + [\lambda, l]. \tag{7}$$

Here the sought function $g(t)$ characterizes the derivative of opening of the pre-fracture band faces:

$$\frac{1+\varkappa_0}{2\mu}\, g(x) = \frac{\partial}{\partial x}\,[v^+(x,0) - v^-(x,0)] \tag{8}$$

(μ is the shear modulus of the plate material).

To find the complex potentials $\Phi_2(z)$ and $\Psi_2(z)$, we present the boundary condition (3) in the form

$$\Phi_2(\tau) + \overline{\Phi_2(\tau)} - [\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)]\, e^{2i\theta} = -\Phi_*(\tau) - \overline{\Phi_*(\tau)} + [\bar{\tau}\Phi_*'(\tau) + \Psi_*(\tau)]\, e^{2i\theta}, \tag{9}$$

where $\Phi_*(\tau) = \Phi_0(\tau) + \Phi_1(\tau)$ and $\Psi_*(\tau) = \Psi_0(\tau) + \Psi_1(\tau)$.

We seek for the functions $\Phi_2(z)$ and $\Psi_2(z)$ in the form [7]

$$\Phi_2(z) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!},$$

$$\Psi_2(z) = \sum_{k=0}^{\infty} \beta_{2k+2}\,\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2}\,\frac{\lambda^{2k+2}S^{(2k+1)}(z)}{(2k+1)!}. \tag{10}$$

The conditions of symmetry about the coordinate axes yield the equalities

$$\operatorname{Im}\alpha_{2k+2} = 0, \qquad \operatorname{Im}\beta_{2k+2} = 0, \qquad k = 0, 1, 2, \ldots\,.$$

Relations (5)–(7) and (10) determine a class of symmetric problems with a periodic distribution of stresses. The condition of a constant main vector of forces acting on the arc connecting two congruent points (with positions that differ by the period ω) in the domain D occupied by the plate material implies that

$$\alpha_0 = \pi^2 \beta_2 \lambda^2/24.$$

The unknown coefficients α_{2k+2} and β_{2k+2} have to be determined from the boundary condition (9). We denote the right side of condition (9) by $f_1(\theta) + i f_2(\theta)$. We assume that the function $f_1(\theta) + i f_2(\theta)$ is decomposed into a Fourier series on the contour $|\tau| = \lambda$. By virtue of symmetry, this series has the form

$$f_1(\theta) + i f_2(\theta) = \sum_{k=-\infty}^{\infty} A_{2k}\, \mathrm{e}^{2ik\theta}, \qquad \mathrm{Im}\, A_{2k} = 0,$$

$$A_{2k} = \frac{1}{2\pi} \int_0^{2\pi} (f_1(\theta) + i f_2(\theta))\, \mathrm{e}^{-2ik\theta}\, d\theta \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{11}$$

Substituting the right side of Eq. (9) into Eqs. (11) and calculating the integrals with the use of the theory of residues, we obtain

$$A_0 = -\frac{1}{2}\,\sigma_0 + \frac{1}{\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{2ny_0}{\rho_1^2}\Big) - \frac{1}{2\omega}\int_{L_1} g(t) f_0(t)\, dt,$$

$$A_2 = \frac{1}{2}\,\sigma_0 - \frac{1}{\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\Big(\frac{\lambda^2 \sin 3\varphi_1}{\rho_1^3} + \frac{\varkappa_0 \sin\varphi_1}{\rho_1} - \frac{\sin 3\varphi_1}{\rho_1}\Big) - \frac{1}{2\omega}\int_{L_1} g(t) f_2(t)\, dt,$$

$$A_{2k} = \frac{1}{\pi h(1+\varkappa_0)}\Big[\sum_{m,n}{}' F_{mn}\Big(\frac{\lambda^{2k}\sin(2k+1)\varphi_1}{\rho_1^{2k+1}} + \frac{(-2)(-3)\cdots(-2k)\lambda^{2k}\sin(2k+1)\varphi_1}{(2k-1)!\rho_1^{2k+1}}$$
$$- \frac{\varkappa_0 \lambda^{2k-2}\sin(2k-1)\varphi_1}{\rho_1^{2k-1}} + \frac{(-2)(-3)\cdots(1-2k)\lambda^{2k-2}\sin(2k+1)\varphi_1}{(2k-2)!\rho_1^{2k-1}}\Big)\Big]$$
$$- \frac{1}{2\omega}\int_{L_1^*} g(t) f_{2k}(t)\, dt \qquad (k = 2, 3, \ldots),$$

$$A_{-2k} = \frac{1}{\pi h(1+\varkappa_0)} \sum_{m,n}{}' F_{mn}\, \frac{\lambda^{2k}\sin(2k+1)\varphi_1}{\rho_1^{2k+1}} - \frac{1}{2\omega}\int_{L_1^*} g(t) f_{-2k}(t)\, dt \qquad (k = 1, 2, \ldots),$$

where

$$\rho_1^2 = (mL)^2 + (ny_0)^2, \qquad \varphi_1 = \arctan\frac{ny_0}{mL},$$

$$f_0(t) = 2\gamma(t), \qquad f_2(t) = -\frac{\lambda^2}{2}\,\gamma^{(2)}(t), \qquad \gamma(t) = \cot\frac{\pi}{\omega}\,t,$$

$$f_{2k}(t) = -\frac{\lambda^{2k}(2k-1)}{(2k)!}\,\gamma^{(2k)}(t) + \frac{\lambda^{2k-2}}{(2k-3)!}\,\gamma^{(2k-2)}(t) \qquad (k = 2, 3, \ldots),$$

$$f_{-2k}(t) = -\frac{\lambda^{2k}}{(2k)!}\,\gamma^{(2k)}(t) \qquad (k = 1, 2, \ldots).$$

The unknown function $g(x)$ and the coefficients α_{2k+2} and β_{2k+2} are determined from the boundary conditions (4) and (9). As the periodicity conditions are satisfied, the system of the boundary conditions (9) degenerates into one functional equation, for instance, on the contour L_0 ($\tau = \lambda\, \mathrm{e}^{i\theta}$), and the system of the boundary conditions (4) degenerates into the boundary condition on the line L_1^*.

To construct equations with respect to the coefficients α_{2k+2} and β_{2k+2} of the functions $\Phi_2(z)$ and $\Psi_2(z)$, we expand these functions into the Laurent series in the neighborhood of the point $z = 0$. Substituting, instead of $\Phi_2(z)$, $\overline{\Phi_2(z)}$, $\Phi_2'(z)$, and $\Psi_2(z)$, their expansions into the Laurent series in the neighborhood of $z = 0$ in the left side of the boundary condition (9) on the contour $z = \lambda\, e^{i\theta}$, substituting the Fourier series (11) instead of the function $f_1(\theta) + if_2(\theta)$ into the right side of Eq. (9), and comparing the coefficients at identical powers of $e^{i\theta}$, we obtain two infinite systems of algebraic equations with respect to the coefficients α_{2k+2} and β_{2k+2}. After some transformations, we obtain an infinite system of linear algebraic equations with respect to α_{2k+2}:

$$\alpha_{2j+2} = \sum_{k=0}^{\infty} a_{j,k}\alpha_{2k+2} + b_j \qquad (j = 0, 1, 2, \ldots). \tag{12}$$

Here

$$b_0 = A_2 - \sum_{k=0}^{\infty} \frac{g_{k+2}\lambda^{2k+4}}{2^{2k+4}}\, A_{-2k-2},$$

$$b_j = A_{2j+2} - \frac{(2j+1)A_0 g_{j+1}\lambda^{2j+2}}{2^{2j+2}K_1} - \sum_{k=0}^{\infty} \frac{(2j+2k+3)g_{j+k+2}\lambda^{2j+2k+4}}{(2j)!(2k+3)!2^{2j+2k+4}}\, A_{-2k-2},$$

$$a_{j,k} = (2j+1)\gamma_{j,k}\lambda^{2j+2k+2}, \qquad K_1 = 1 - \frac{\pi^2}{12}\,\lambda^2, \qquad g_j = 2\sum_{m=1}^{\infty} \frac{1}{m^{2j}},$$

$$\gamma_{0,0} = \frac{3}{8}\, g_2\lambda^2 + \sum_{i=1}^{\infty} \frac{(2i+1)g_{i+1}^2\lambda^{4i+2}}{2^{4i+4}}, \qquad \gamma_{j,k} = -\frac{(2j+2k+2)!g_{k+j+1}}{(2j+1)!(2k+1)!2^{2j+2k+2}} + \frac{(2j+2k+4)!g_{j+k+2}\lambda^2}{(2j+2)!(2k+2)!2^{2j+2k+4}}$$

$$+ \sum_{i=0}^{\infty} \frac{(2j+2i+1)!(2k+2i+1)!g_{j+i+1}g_{k+i+1}\lambda^{4i+2}}{(2j+1)!(2k+1)!(2i+1)!(2i)!2^{2j+2k+4i+4}} + b_{j,k},$$

$$b_{0,k} = 0, \qquad b_{j,0} = 0,$$

$$b_{j,k} = \frac{g_{j+1}g_{k+1}\lambda^2}{2^{2j+2k+4}}\Big(1 + \frac{2K_2\lambda^2}{K_1}\Big), \quad j = 1, 2, \ldots, \quad k = 1, 2, \ldots, \qquad K_2 = \frac{\pi^2}{24}.$$

The constants β_{2k+2} are determined from the following relations:

$$\beta_2 = \frac{1}{K_1}\Big(- A_0 + 2\sum_{k=0}^{\infty} \frac{g_{k+1}\lambda^{2k+2}}{2^{2k+2}}\, \alpha_{2k+2}\Big),$$

$$\beta_{2j+4} = (2j+3)\alpha_{2j+2} + \sum_{k=0}^{\infty} \frac{(2j+2k+3)!g_{j+k+2}\lambda^{2j+2k+4}}{(2j+2)!(2k+1)!2^{2j+2k+4}}\, \alpha_{2k+2} - A_{-2j-2}. \tag{13}$$

Requiring that functions (5) satisfy the boundary condition (4) and applying some transformations, we obtain a singular integral equation with respect to the function $g(x)$:

$$\frac{1}{\omega}\int_{L_1} g(t)\cot\frac{\pi}{\omega}\,(t-x)\,dt + H(x) = \sigma_{\text{yield}}. \tag{14}$$

Here $H(x) = \Phi_s(x) + \overline{\Phi_s(x)} + x\Phi_s'(x) + \Psi_s(x)$, $\Phi_s(x) = \Phi_0(x) + \Phi_2(x)$, and $\Psi_s(x) = \Psi_0(x) + \Psi_2(x)$.

The singular integral equation (14) and systems (12) and (13) contain unknown point forces F_{mn} ($m = 1, 2, \ldots$; $n = 1, 2, \ldots$). According to Hooke's law, the value of the point force F_{mn} acting onto each point of attachment from the side of the stringer is

$$F_{mn} = \frac{E_s A_s}{2y_0 n}\,\Delta v_{m,n} \qquad (m = 1, 2, \ldots; \quad n = 1, 2, \ldots).$$

Here E_s is Young's modulus of the stringer material, A_s is the cross-sectional area of the stringer, $2y_0n$ is the distance between the attachment points, and $\Delta v_{m,n}$ is the dimensionless displacement of the considered attachment points, which is equal to the elongation of the corresponding segment of the stringer.

Let us denote the radius of the attachment points (attachment area) by a_0. We use a natural assumption that the dimensionless elastic displacement of the points $z = mL + i(ny_0 - a_0)$ and $z = mL - i(ny_0 - a_0)$ in the considered problem of the elasticity theory equals the dimensionless displacement of the attachment points $\Delta v_{m,n}$. This additional condition of compatibility of displacements allows us to find the solution of the problem posed above.

Using the complex potentials (5)–(7), (10) and the Kolosov–Muskhelishvili formulas [8], we find the dimensionless displacement $\Delta v_{m,n}$:

$$\Delta v_{p,r} = \Delta v_{p,r}^{(0)} + \Delta v_{p,r}^{(1)} + \Delta v_{p,r}^{(2)}. \tag{15}$$

Here

$$\Delta v_{p,r}^{(0)} = \frac{1}{2\pi(1+\varkappa_0)\mu h} \sum_{m,n}{}' F_{mn}\Big(\varkappa_0 \ln \frac{(p-m)^2L^2 + a_0^2}{(p-m)^2L^2 + c^2}$$

$$+ \frac{2(r-n)y_0c[2p(p-m)L^2 + a_0c]}{[(p-m)^2L^2 + c^2][(p-m)^2L^2 + a_0^2]}\Big) + \frac{\sigma_0}{4\mu}(1+\varkappa_0)(ry_0 - a_0),$$

$$\Delta v_{p,r}^{(1)} = \frac{1+\varkappa_0}{\mu}\Big\{\frac{1}{2\omega}\int_{L_1} g(t)\Big[\arctan\Big(\cot\frac{\pi}{\omega}(t-pL)\tanh\frac{\pi}{\omega}c\Big) - \arctan\Big(\tan\frac{\pi pL}{\omega}\tanh\frac{\pi}{\omega}c\Big)\Big]\,dt\Big\}$$

$$-\frac{c}{\mu}\frac{1}{2\omega}\int_{L_1} g(t)\,\frac{\sin^2\alpha_1(\cosh^2\alpha_1 + \sinh^2\alpha_1)}{\sin^2\alpha_1\cosh^2\alpha_1 + \cos^2\alpha_1\sinh^2\alpha_1}\,dt,$$

$$\Delta v_{p,r}^{(2)} = \frac{1}{\mu}\Big((\varkappa_0 - 1)(ry_0 - a_0)a_0 + (1+\varkappa_0)\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\sin(2k+1)\alpha}{(2k+1)\rho_2^{2k+1}}$$

$$+ (\varkappa_0 - 1)\sum_{k=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{r_{j,k}}{2j+1}\rho_2^{2j+1}\sin(2j+1)\alpha - \sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\sin(2k+1)\alpha}{(2k+1)\rho_2^{2k+1}}$$

$$-\sum_{k=0}^{\infty}\beta_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{r_{j,k}}{2j+1}\rho_2^{2j+1}\sin(2j+1)\alpha$$

$$+\sum_{k=0}^{\infty}(2k+2)\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{2j+2k+2}{2j+1}r_{j,k}\rho_2^{2j+1}\sin(2j+1)\alpha\Big),$$

$$c = (r-n)y_0 - a_0, \qquad \alpha_1 = \frac{\pi}{\omega}(t-pL), \qquad \alpha = \arctan\frac{ry_0 - a_0}{pL},$$

$$\rho_2^2 = (pL)^2 + (ry_0 - a_0)^2, \qquad r_{j,k} = \frac{(2j+2k+1)!g_{j+k+1}}{(2j)!(2k+1)!2^{2j+2k+2}}, \qquad r_{0,0} = 0.$$

The sought value of the force F_{mn} is determined by Eqs. (15) from the system

$$F_{pr} = \frac{E_sA_s}{2y_0r}\Delta v_{p,r} \qquad (p = 1,2,\ldots;\ r = 1,2,\ldots), \tag{16}$$

which degenerates into one infinite algebraic system by virtue of problem periodicity.

The singular integral equation (14) and also systems (12), (13), and (16) are the basic resolving equations of the problem, which allow the function $g(x)$, the coefficients α_{2k} and β_{2k}, and the forces F_{pr} ($p = 1,2,\ldots;$

$r = 1, 2, \ldots$) to be determined. Knowing the functions $\Phi_2(z)$, $\Psi_2(z)$, and $g(x)$, and also the values of F_{pr}, we can find the stress–strain state of the reinforced plate with pre-fracture bands.

Numerical Solution of the Problem and Analysis of Results. Using the expansion

$$\frac{\pi}{\omega}\cot\frac{\pi}{\omega}z = \frac{1}{z} - \sum_{j=0}^{\infty} g_{j+1}\frac{z^{2j+1}}{\omega^{2j+2}},$$

we can convert Eq. (14) to the standard form

$$\frac{1}{\pi}\int_{L_1}\frac{g(t)\,dt}{t-x} + \frac{1}{\pi}\int_{L_1} g(t)K(t-x)\,dt = \sigma_{\text{yield}}, \tag{17}$$

where

$$K(t) = -\sum_{j=0}^{\infty} g_{j+1}\frac{t^{2j+1}}{\omega^{2j+2}}.$$

With allowance for $g(x) = -g(-x)$, we convert the integral equation (17) to a form more convenient for finding the approximate solution:

$$\frac{2}{\pi}\int_{\lambda_1}^{1}\frac{\xi p(\xi)\,d\xi}{\xi^2-\xi_0^2} + \frac{1}{\pi}\int_{\lambda_1}^{1} K_0(\xi,\xi_0)p(\xi)\,d\xi + H(\xi_0) = \sigma_{\text{yield}}. \tag{18}$$

Here

$$K_0(\xi,\xi_0) = K(\xi-\xi_0) + K(\xi+\xi_0), \qquad p(\xi) = g(t), \qquad \xi = t/l, \qquad \xi_0 = x/l,$$

$$\lambda_1 = \lambda/l, \quad \lambda_1 \leqslant \xi_0 \leqslant 1, \qquad H(\xi_0) = \Phi_s(\xi_0 l) + \overline{\Phi_s(\xi_0 l)} + \xi_0 l\Phi_s'(\xi_0 l) + \Psi_s(\xi_0 l).$$

We replace the variables as

$$\xi^2 = u = \frac{1-\lambda_1^2}{2}(\tau+1) + \lambda_1^2, \qquad \xi_0^2 = u_0 = \frac{1-\lambda_1^2}{2}(\eta+1) + \lambda_1^2.$$

The interval of integration $[\lambda_1, 1]$ transforms to the interval $[-1, 1]$, and the transformed Eq. (18) acquires the standard form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(\tau)\,d\tau}{\tau-\eta} + \frac{1}{\pi}\int_{-1}^{1} p(\tau)B(\eta,\tau)\,d\tau + H_*(\eta) = \sigma_{\text{yield}}, \tag{19}$$

where

$$p(\tau) = p(\xi), \qquad H_*(\eta) = H(\xi_0), \qquad B(\eta,\tau) = -\frac{1-\lambda_1^2}{2}\sum_{j=0}^{\infty} g_{j+1}\Big(\frac{l}{2}\Big)^{2j+2} u_0^j A_j,$$

$$A_j = 2j+1 + \frac{(2j+1)(2j)(2j-1)}{1\cdot 2\cdot 3}\Big(\frac{u}{u_0}\Big) + \ldots + \frac{(2j+1)(2j)(2j-1)\cdots[2j+1-(2j+1-1)]}{1\cdot 2\cdot 3\cdots(2j+1)}\Big(\frac{u}{u_0}\Big)^j.$$

To replace the singular integral equation by a system of algebraic equations, we use the method of the direct solution of singular integral equations [6–8]. In addition to the singularity in the Cauchy kernel, the integral equation (19) has a motionless singularity at the point where the pre-fracture band reaches the surface of the circular hole. In this case, the function $g(x)$ at $x = \pm\lambda$ has a singularity other than the root singularity. The character of this singularity can be determined by analyzing the singular integral equation (19) [9]. In the case considered, we have the integral $\int_{\lambda}^{l} g(t)\,dt = C \neq 0$. The constant C is expressed via the opening of the pre-fracture band on the surface of the circular hole and has to be determined after solving the singular integral equation.

In the case considered, we should apparently use the method of solving the integral equation constructed on the basis of the Gauss–Jacobi quadrature formula. As the expressions for the functions $B(\eta,\tau)$ and $H_*(\eta)$ are too cumbersome, it is difficult to determine the singularity of the function $p(\eta)$ at the ends of the interval (at the points $x = \pm\lambda$). In addition, it should be noted that a certain gain in terms of convergence provided by the refined method is spoiled because of the cumbersome formulas for the matrix coefficients of the system. We use another method for the numerical solution of integral equations of the type (19). The efficiency of this method was verified by solving numerous problems [10–12]. As the stresses in the reinforced plate are limited, the solution of the singular integral equation (19) has to be sought in a class of functions bounded everywhere. The solution can be presented in the form

$$p(\eta) = p_0(\eta)\sqrt{1-\eta^2},$$

where $p_0(\eta)$ is a new unknown bounded function on the segment $[-1, 1]$.

Using the quadrature formulas [7, 10], we can reduce the integral equation (19) to a system of $M+1$ algebraic equations

$$\sum_{m=1}^{M} \frac{p_0(\tau_m)}{M+1} \sin^2 \frac{\pi m}{M+1} \Big(\frac{1}{\tau_m - \eta_r} + B(\tau_m, \eta_r) \Big) = \pi[\sigma_{\text{yield}} - H_*(\eta_r)]$$
$$(r = 1, 2, \ldots, M+1), \tag{20}$$

where

$$\tau_m = \cos \frac{\pi m}{M+1} \quad (m = 1, 2, \ldots, M), \qquad \eta_r = \cos \Big(\frac{2r-1}{2(M+1)}\, \pi \Big) \quad (r = 1, 2, \ldots, M+1).$$

The resultant algebraic system (20) of $M+1$ equations for determining the unknowns $p_0(\tau_1)$, $p_0(\tau_2)$, ... , $p_0(\tau_M)$, and l/λ satisfies an additional condition at which there exists a solution in a class of functions bounded everywhere (see [9]).

As the size of the pre-fracture zone is unknown, system (20) is nonlinear. The systems of equations with respect to the unknowns α_{2k+2}, β_{2k+2}, $p_0(\tau_m)$ $(m = 1, 2, \ldots, M)$, F_{mn} $(m = 1, 2, \ldots,\ n = 1, 2, \ldots)$ and l/λ with a given external tensile load allow us to determine the stress–strain state of a perforated plate with pre-fracture bands. As the quantity l is unknown, the united system (12), (13), (16), (20) is nonlinear. Taking into account that the tensile load σ_0 is a linear term of these equations, we use the inverse method of solving this problem. We assume that the length of the pre-fracture zone l is given, and the corresponding external load is found in the course of solving system (12), (13), (16), (20). Such an approach implies solving a linear algebraic system for each value of l. The calculations were performed for the following geometric parameters of the reinforced plate: $\nu = 0.3$, $\varepsilon_1 = a_0/L = 0.01$, $\varepsilon = y_0/L = 0.15$, 0.25, and 0.50, $E = 7.1 \cdot 10^4$ MPa (V95 alloy), $E_s = 11.5 \cdot 10^4$ MPa (aluminum–steel composite), and $A_s/(y_0 h) = 1$. The number of stringers and attachment points was assumed to be 14 and 10, respectively. We used $M = 30$, which corresponds to splitting the integration interval into 30 Chebyshev nodes. Each of the infinite systems (12) and (13) was truncated to five equations, and the unknown coefficients β_{2k} were eliminated from the remaining equations with the help of Eqs. (13). Figure 2 shows the length of the pre-fracture band $d = (l-\lambda)/L$ as a function of the dimensionless external load $\sigma_0/\sigma_{\text{yield}}$ for $\varepsilon = 0.25$. Curves 1–5 were plotted until the values of d corresponding to crack emergence were reached.

Using the solution of the problem of plastic strains in the band, we calculate the displacements of the points on the pre-fracture band faces:

$$v(x, 0) = -\frac{1+\varkappa_0}{2\mu} \int_{-l}^{x} g(x)\, dx$$

[the function $g(x)$ was determined by Eq. (8)]. For $x = \pm\lambda$, we have

$$v(-\lambda, 0) = -\frac{1+\varkappa_0}{2\mu} \int_{-l}^{-\lambda} g(x)\, dx.$$

Replacing the integral by the sum, we obtain

$$v(-\lambda, 0) = -\frac{1+\varkappa_0}{2\mu}\, \frac{\pi(l-\lambda)}{M} \sum_{m=1}^{M} g(t_m).$$

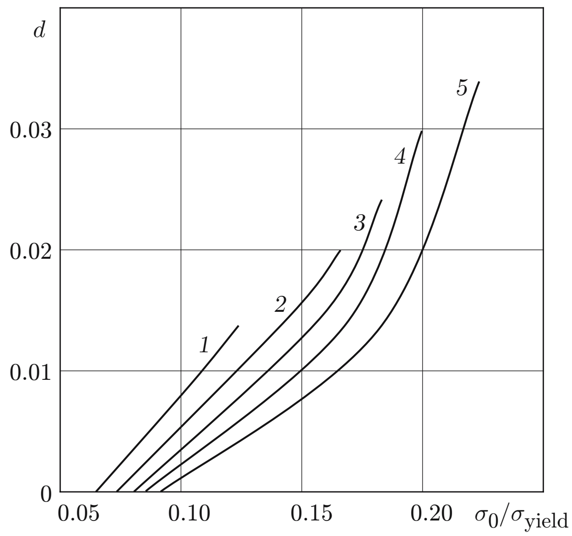

Fig. 2

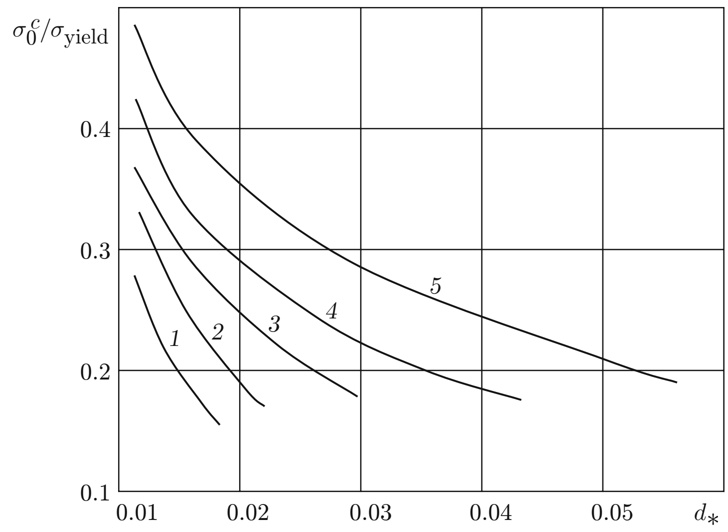

Fig. 3

Fig. 2. Dimensionless length of the pre-fracture band d versus the dimensionless external load $\sigma_0/\sigma_{\rm yield}$ for $\varepsilon = 0.25$ and different values of the hole radius: $\lambda = 0.6$ (1), 0.5 (2), 0.4 (3), 0.3 (4), and 0.2 (5).

Fig. 3. Critical external tensile load $\sigma_0^c/\sigma_{\rm yield}$ versus the dimensionless critical length of the pre-fracture band d_* for $\varepsilon = 0.15$ and different values of the hole radius (notation the same as in Fig. 2).

To determine the critical equilibrium state in which the crack emerges, we use condition (2). Thus, the following condition determines the critical state of the tensile load σ_0 for the reinforced plate:

$$v(-\lambda, 0) = \delta_c. \tag{21}$$

The solution of the algebraic system (12), (13), (16), (20), (21) allows us to determine the critical value of the external tensile load and the size of the pre-fracture band in the critical state in which cracks emerge.

Figure 3 shows the critical external tensile load $\sigma_0^c/\sigma_{\rm yield}$ as a function of the critical length of the pre-fracture band $d_* = (l_* - \lambda)/L$ for $\varepsilon = 0.15$, $\delta_c^*/(l_* - \lambda) = 0.5$, and $\delta_c^* = \pi\delta_c\mu/[(1 + \varkappa_0)\sigma_{\rm yield}]$.

The analysis of the model of crack emergence in a reinforced plate attenuated by a periodic system of circular holes reduces to a parametric study of the algebraic system (12), (13), (16), (20) and the criterion of crack emergence (21) for different parameters of the reinforced plate (mechanical characteristics of materials and geometric parameters of the reinforced plate).

In conclusion, it should be noted that the constructed model of crack formation in a reinforced plate is based on the concept of the dislocation mechanism of crack nucleation, which was experimentally validated (see, e.g., [2]). The calculation model proposed can fill the gap between the microscopic theory of dislocations and the phenomenological theory of strength.

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